Interpreters

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Finite Maps

$\bullet$	empty map, with $\varnothing$ as its domain
$m(k)$	mapping of key k in map m
$m[k \mapsto v]$	extension of map m to also map key k to value v

$$\frac{}{m[k \mapsto v](k) = v} \qquad \frac{k_1 \neq k_2}{m[k_1 \mapsto v](k_2) = m(k_2)}$$

Interpretation

$$\begin{array}{rcl} \text{Constants} & n & \in & \mathbb{N} \\ \text{Variables} & x & \in & \mathsf{Strings} \\ \text{Expressions} & e & ::= & n \mid x \mid e + e \mid e \times e \end{array}$$

$$\begin{array}{rcl} [\![n]\!]v & = & n \\ [\![x]\!]v & = & v(x) \\ [\![e_1 + e_2]\!]v & = & [\![e_1]\!]v + [\![e_2]\!]v \\ [\![e_1 \times e_2]\!]v & = & [\![e_1]\!]v \times [\![e_2]\!]v \end{array}$$

https://semanticsarchive.net/Archive/TE0Y2RkN/.denotation_brackets.pdf

Substitution

$$
\begin{array}{rcl}
[e/x]n & = & n \\
[e/x]x & = & e \\
[e/x]y & = & y, \text{ when } y \neq x \\
[e/x](e_1 + e_2) & = & [e/x]e_1 + [e/x]e_2 \\
[e/x](e_1 \times e_2) & = & [e/x]e_1 \times [e/x]e_2
\end{array}
$$

THEOREM 4.1. *For all* e, e', x, *and* v, $[\![[e'/x]e]\!]v = [\![e]\!](v[x \mapsto [\![e']\!]v])$.

A Stack Machine

$$
\begin{array}{rcl}
\text{Instructions} \quad i & ::= & \mathsf{PushConst}(n) \mid \mathsf{PushVar}(x) \mid \mathsf{Add} \mid \mathsf{Multiply} \\
\text{Programs} \quad \bar{i} & ::= & \cdot \mid i; \bar{i}
\end{array}
$$

$$
\begin{array}{rcl}
[\![\mathsf{PushConst}(n)]\!](v, s) & = & n \rhd s \\
[\![\mathsf{PushVar}(x)]\!](v, s) & = & v(x) \rhd s \\
[\![\mathsf{Add}]\!](v, n_2 \rhd n_1 \rhd s) & = & (n_1 + n_2) \rhd s \\
[\![\mathsf{Multiply}]\!](v, n_2 \rhd n_1 \rhd s) & = & (n_1 \times n_2) \rhd s
\end{array}
$$

A Stack Machine

$$
\begin{aligned}
\lfloor n \rfloor &= \mathsf{PushConst}(n) \\
\lfloor x \rfloor &= \mathsf{PushVar}(x) \\
\lfloor e_1 + e_2 \rfloor &= \lfloor e_1 \rfloor \bowtie \lfloor e_2 \rfloor \bowtie \mathsf{Add} \\
\lfloor e_1 \times e_2 \rfloor &= \lfloor e_1 \rfloor \bowtie \lfloor e_2 \rfloor \bowtie \mathsf{Multiply}
\end{aligned}
$$

THEOREM 4.2. $[\![\lfloor e \rfloor]\!](v, \cdot) = [\![e]\!]v.$

Imperative Language

$$
\begin{array}{rccl}
\text{Constants} & n & \in & \mathbb{N} \\
\text{Variables} & x & \in & \mathsf{Strings} \\
\text{Expressions} & e & ::= & n \mid x \mid e + e \mid e \times e \\
\text{Command} & c & ::= & \mathsf{skip} \mid x \leftarrow e \mid c ; c \mid \mathsf{repeat}\ e\ \mathsf{do}\ c\ \mathsf{done}
\end{array}
$$

$$
\begin{array}{rcl}
f^0 & = & \mathsf{id} \\
f^{n+1} & = & f^n \circ f
\end{array}
$$

$$
\begin{array}{rcl}
[\![\mathsf{skip}]\!]v & = & v \\
[\![x \leftarrow e]\!]v & = & v[x \mapsto [\![e]\!]v] \\
[\![c_1 ; c_2]\!]v & = & [\![c_2]\!]([\![c_1]\!]v) \\
[\![\mathsf{repeat}\ e\ \mathsf{do}\ c\ \mathsf{done}]\!]v & = & [\![c]\!]^{[\![e]\!]v}(v)
\end{array}
$$

Loop Unrolling

$$
\begin{aligned}
{}^{0}c &= \mathsf{skip} \\
{}^{n+1}c &= c; {}^{n}c
\end{aligned}
$$

$$
\begin{aligned}
|\mathsf{skip}| &= \mathsf{skip} \\
|x \leftarrow e| &= x \leftarrow e \\
|c_1; c_2| &= |c_1|; |c_2| \\
|\mathsf{repeat}\ n\ \mathsf{do}\ c\ \mathsf{done}| &= {}^{n}|c| \\
|\mathsf{repeat}\ e\ \mathsf{do}\ c\ \mathsf{done}| &= \mathsf{repeat}\ e\ \mathsf{do}\ |c|\ \mathsf{done}
\end{aligned}
$$

THEOREM 4.4. $[\![|c|]\!]v = [\![c]\!]v$.