Evaluation Contexts

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Why?

- *Modularly* add features to the core language
- Don't need to redo proofs
  - ….with advanced mechanisation
  - In our course, we still need to do a bit of work.

STLC

$$\text{Types} \quad \tau \quad ::= \quad \mathbb{N} \mid \tau \to \tau$$

$$\begin{array}{rccl}
\text{Variables} & x & \in & \mathsf{Strings} \\
\text{Numbers} & n & \in & \mathbb{N} \\
\text{Expressions} & e & ::= & n \mid e + e \mid x \mid \lambda x.\ e \mid e\ e \\
\text{Values} & v & ::= & n \mid \lambda x.\ e
\end{array}$$

Key Reductions

$$\frac{}{(\lambda x.\ e)\ v \to [v/x]e} \qquad \frac{}{n + m \to n{+}m}$$

Administrative Reductions

$$\frac{e_1 \to e_1'}{e_1\ e_2 \to e_1'\ e_2} \qquad \frac{e_2 \to e_2'}{v\ e_2 \to v\ e_2'}$$

$$\frac{e_1 \to e_1'}{e_1 + e_2 \to e_1' + e_2} \qquad \frac{e_2 \to e_2'}{v + e_2 \to v + e_2'}$$

STLC using Evaluation Contexts

$$\text{Types} \quad \tau \quad ::= \quad \mathbb{N} \mid \tau \to \tau$$

$$\begin{array}{rccl} \text{Variables} & x & \in & \mathsf{Strings} \\ \text{Numbers} & n & \in & \mathbb{N} \\ \text{Expressions} & e & ::= & n \mid e + e \mid x \mid \lambda x.\, e \mid e\ e \\ \text{Values} & v & ::= & n \mid \lambda x.\, e \end{array}$$

$$\text{Evaluation contexts} \quad C \quad ::= \quad \square \mid C\ e \mid v\ C \mid C + e \mid v + C$$

Key Reductions

$$\frac{}{(\lambda x.\, e)\ v \to_0 [v/x]e} \qquad \frac{}{n + m \to_0 n{+}m}$$

Administrative Reductions

$$\frac{e \to_0 e'}{C[e] \to C[e']}$$

Preservation Proof

Lemma **preservation0**

$$\textit{If } e_1 \to_0 e_2 \ \textit{and} \vdash e_1 : \tau, \ \textit{then} \vdash e_2 : \tau.$$

Lemma **preservation'**

$$\textit{If } e_1 \to_0 e_2 \ \textit{and} \vdash C[e_1] : \tau, \ \textit{then} \vdash C[e_2] : \tau.$$

Lemma **preservation**

$$\textit{If } e_1 \to e_2 \ \textit{and} \vdash e_1 : \tau, \ \textit{then} \vdash e_2 : \tau.$$

Product Types

$$\begin{array}{rccl} \text{Expressions} & e & ::= & \ldots \mid (e, e) \mid \pi_1(e) \mid \pi_2(e) \\ \text{Values} & v & ::= & \ldots \mid (v, v) \\ \text{Contexts} & C & ::= & \ldots \mid (C, e) \mid (v, C) \mid \pi_1(C) \mid \pi_2(C) \\ \text{Types} & \tau & ::= & \ldots \mid \tau \times \tau \end{array}$$

Typing Rules

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (e_1, e_2) : \tau_1 \times \tau_2} \quad \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \pi_1(e) : \tau_1} \quad \frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \pi_2(e) : \tau_2}$$

Key Reductions

$$\frac{}{\pi_1((v_1, v_2)) \rightarrow_0 v_1} \quad \frac{}{\pi_2((v_1, v_2)) \rightarrow_0 v_2}$$

Administrative Reductions

No new rules!

Sum Types

$$
\begin{array}{rrcl}
\text{Expressions} & e & ::= & \ldots \mid \mathsf{inl}(e) \mid \mathsf{inr}(e) \mid (\mathsf{match}\ e\ \mathsf{with}\ \mathsf{inl}(x) \Rightarrow e \mid \mathsf{inr}(x) \Rightarrow e) \\
\text{Values} & v & ::= & \ldots \mid \mathsf{inl}(v) \mid \mathsf{inr}(v) \\
\text{Contexts} & C & ::= & \ldots \mid (\mathsf{match}\ C\ \mathsf{with}\ \mathsf{inl}(x) \Rightarrow e \mid \mathsf{inr}(x) \Rightarrow e) \\
\text{Types} & \tau & ::= & \ldots \mid \tau + \tau
\end{array}
$$

Typing Rules

$$
\frac{\Gamma \vdash e : \tau_1}{\Gamma \vdash \mathsf{inl}(e) : \tau_1 + \tau_2} \qquad \frac{\Gamma \vdash e : \tau_2}{\Gamma \vdash \mathsf{inr}(e) : \tau_1 + \tau_2}
$$

$$
\frac{\Gamma \vdash e : \tau_1 + \tau_2 \quad \Gamma, x_1 : \tau_1 \vdash e_1 : \tau \quad \Gamma, x_2 : \tau_2 \vdash e_2 : \tau}{\Gamma \vdash (\mathsf{match}\ e\ \mathsf{with}\ \mathsf{inl}(x_1) \Rightarrow e_1 \mid \mathsf{inr}(x_2) \Rightarrow e_2) : \tau}
$$

Key Reductions

$$
\frac{}{(\mathsf{match}\ \mathsf{inl}(v)\ \mathsf{with}\ \mathsf{inl}(x_1) \Rightarrow e_1 \mid \mathsf{inr}(x_2) \Rightarrow e_2) \rightarrow_0 [v/x_1]e_1}
$$

$$
\frac{}{(\mathsf{match}\ \mathsf{inr}(v)\ \mathsf{with}\ \mathsf{inl}(x_1) \Rightarrow e_1 \mid \mathsf{inr}(x_2) \Rightarrow e_2) \rightarrow_0 [v/x_2]e_2}
$$

Administrative Reductions

No new rules!

Exceptions

$$\begin{array}{rccl} \text{Expressions} & e & ::= & \ldots \mid \mathsf{throw}(e) \mid (\mathsf{try}\ e\ \mathsf{catch}\ x \Rightarrow e) \\ \text{Contexts} & C & ::= & \ldots \mid \mathsf{throw}(C) \mid (\mathsf{try}\ C\ \mathsf{catch}\ x \Rightarrow e) \end{array}$$

Typing Rules

$$\frac{\Gamma \vdash e : \mathbb{N}}{\Gamma \vdash \mathsf{throw}(e) : \tau} \qquad \frac{\Gamma \vdash e : \tau \quad \Gamma, x_1 : \mathbb{N} \vdash e_1 : \tau}{\Gamma \vdash (\mathsf{try}\ e\ \mathsf{catch}\ x_1 \Rightarrow e_1) : \tau}$$

Key Reductions

$$\frac{}{(\mathsf{try}\ v\ \mathsf{catch}\ x \Rightarrow e) \to_0 v} \qquad \frac{}{(\mathsf{try}\ \mathsf{throw}(v)\ \mathsf{catch}\ x \Rightarrow e) \to_0 [v/x]e}$$

$$\frac{}{C^-[\mathsf{throw}(v)] \to_0 \mathsf{throw}(v)}$$

No catch nor is a hole

Invariant $I(e) = e \text{ is a value} \vee (\exists n : \mathbb{N}.\ e = \mathsf{throw}(n)) \vee (\exists e'.\ e \to e')$